

Experimental study of Bernoulli's equation with losses

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The purpose of this work is to present a simple and inexpensive experiment to study the drainage of a cylindrical vessel. The experimental set up consists of a transparent cylinder and a web cam or a digital camera connected to a computer. The model proposed to explain the results makes use of Bernoulli's equation for real flows that includes energy losses. The experiment allows us to test thoroughly the implication of the model and to extract the relevant parameter associated with the energy losses. The experimental results can be nicely explained by the model, which is a generalization of Torricelli's expression, and clearly illustrate the utility of the extended Bernoulli equation for real fluids.

I. - Introduction

The Bernoulli equation including losses is an useful and important expression of wide application in many branches of science and engineering.^{1,2,3} General forms of Bernoulli's equation, also known as the extended Bernoulli's equation, valid for viscous fluids, have been discussed previously in this journal⁴. Nonetheless, there are relatively few experiments,^{5,6} accessible to beginner and intermediate students that illustrate its use and applications. In this work we present a conceptually simple and inexpensive experiment to study the drainage of a cylindrical vessel that, we believe, clearly illustrates an application of Bernoulli's equation for real liquids. It is essentially a recreation of Torricelli's experiment, profiting from the advantages of new technologies. The experimental set up consists essentially of a transparent cylinder and a web cam connected to a computer. In the first part of this study we present a model based on the Bernoulli equation for real flows. Then we present the basic characteristic of the experiment and the experimental results. The experiment allows us to thoroughly test the implications of the model and to extract the relevant parameter associated with the energy losses. Within this context, we find this experiment pedagogically useful to introduce the Bernoulli equation for real flows to beginner students. The experiment involves concepts that are relatively simple to discuss theoretically, the physics is easy to visualize and it is quite straightforward to quantitatively test the implication of the model.

II. - Theoretical considerations

For Newtonian fluids the shear stress is proportional to the velocity gradient. Therefore the velocity of the fluid on the surface of a solid must be zero, otherwise the velocity gradient and the shear stress will be infinite. Only for an ideal fluid, that is a

fluid with zero viscosity, $\eta=0$, is it possible to have finite velocity on the surface of a solid. Therefore, when a real fluid flows through the interior of a tube or between two surfaces, there are two effects that are a consequence of the viscosity. The velocity profile presents a maximum at the center of the tube. The other important consequence of the viscosity is the nonconservation of mechanical energy in the system or the presence of energy losses.

In its simplest form, the Bernoulli equation is basically a statement of the conservation of mechanical energy per unit of volume along a stream line,^{1,2}

$$\frac{1}{2}v_1^2 + \frac{P_1}{\rho_1} + g z_1 = \frac{1}{2}v_2^2 + \frac{P_2}{\rho_2} + g z_2. \quad (1)$$

Indexes 1 and 2 refer to two different points in the flow, 1 being upstream of 2, v is the local velocity of the fluid, g represents the local acceleration of gravity, P the pressure and z the vertical height of the point.

In the presence of viscosity, Bernoulli's equation becomes an expression of the energy balance, often expressed in terms of energy per unit of volume or pressure (or energy per unit of weight or head) between two points in the flow of fluid. The Bernoulli equation for a steady flow of real fluids in a pipe can be written in the following form:^{1,2,3,4}

$$\frac{1}{2}\alpha_1 u_1^2 + \frac{P_1}{\rho_1} + g z_1 = \frac{1}{2}\alpha_2 u_2^2 + \frac{P_2}{\rho_2} + g z_2 + \Delta w_{loss}. \quad (2)$$

Here the average velocity, u , of the flow along the tube is defined in terms of the flux Q as:

$$Q = \iint_S \vec{v} \cdot d\vec{S} = u \cdot S. \quad (3)$$

Where S is the area of the normal cross section of the flow (or pipe) and v is the local velocity. Expression (3) can be regarded as the definition of the average velocity u . An important physical consequence of the presence of shear stress (viscosity) against the walls the tube, as mentioned above, is that the velocity profile of the flow across the tube will no longer be constant. The kinetic energy coefficients α_i in Eq. (2) represent the ratio between actual kinetic energy that flows through a normal cross section and the kinetic energy of the same flux, but with uniform velocity profile equal to u , more specifically α is defined as^{1,3}

$$\alpha = \iint_S v^3 \cdot dS / u^3 \cdot S, \quad (4)$$

thus, the term $\frac{1}{2} \alpha u^2$, in equation (2), represents the kinetic energy flow per unit of mass across a surface normal to the pipe. Only for a uniform profile α is equal to 1, for any non uniform profile, it follows straightforwardly from Eqs.(3) and (4) that $\alpha > 1$ and for a parabolic profile of velocity (laminar flow) $\alpha=2$.

For real viscous fluids, mechanical energy is dissipated in the viscous boundary layer along the pipe walls (pipe friction) and changes in velocity profiles in entries and exits (minor losses). The term Δw_{loss} in expression (2) represents these energy losses between the points 1 and 2. Usually the presence of restrictions in the flow leads to the formation turbulences, which are the main source of energy losses. There is not an "ab-initio" general expression for these types of losses. Nonetheless, different expressions that describe these energy losses can be obtained from dimensional analysis for particular cases.^{1,2} The relevant constants associated with these expressions need to be determined

from experimental studies. Also experiments indicate that whenever there is a restriction in the flow, produced by bends, tees, valves, and the like, a drop in pressure appears across the component. This pressure drop depends on the geometry of the restriction and has been observed to be proportional to the flowrate squared,^{1,2} i.e. $\Delta P \propto Q^2 \propto u^2$. As we mentioned above, the presence of obstructions in the flow, usually leads to the formation of turbulent wakes. These changes in the flow pattern produce large transfer of momentum from the originally regular flow to the eddy currents and are also associated to a significant increase in the entropy of the system. This entropy increase (ΔS) will require an additional removal of mechanical energy from the flow, $\Delta Q = T \cdot \Delta S$. Since the complexity of turbulence falls far beyond the scope of the present study, we refer the reader to a comprehensive review of the literature on turbulence recently published in this journal.⁷ In this work, we will make the ansatz, that Δw_{loss} can be expressed as the sum of two terms, one dependent on the square of the average velocity and one independent of the velocity, and let the experimental results falsify our hypothesis. The term proportional to u^2 is often referred to as minor loss when it refers to losses that occur at entries, exits, bend, orifice, change of diameter, etc.

In particular, if we apply Bernoulli's equation including loss to our system, figure 1, taking into account that the pressure on the free surface and at exit orifice are the atmospheric pressure P_0 , from Eq. (2) we have:

$$\frac{1}{2g} u_2^2 + \frac{1}{2g} k u_2^2 + \Delta Z = h + \frac{\alpha_1}{2g} u_1^2. \quad (5)$$

Here the second term of the first member represents the energy loss term proportional to u^2 , the constant k is often referred to as the coefficient of minor losses at the exit orifice, point 2 in our case.^{1,2} The quantity ΔZ represents the part of the energy loss that is independent of velocity. As justified in appendix A, the velocity coefficient α_2 is equal to 1 for the exit jet through the orifice. The variable h ($=z_1 - z_2$) represents the height of the free surface relative to the position of the exit orifice.

The velocities u_1 and u_2 are related by the continuity equation (conservation of mass). For an incompressible fluid, ($\rho_1 = \rho_2$) we have:

$$d_2^2 \cdot C_v \cdot u_2 = d_1^2 \cdot u_1. \quad (6)$$

Here d_1 and d_2 represent the diameters of the vessel and the orifice respectively. C_v is the coefficient of *vena contracta*,^{1,2,3} that is related to the fact that the cross section of the exit jet in general is smaller than that of the exit orifice. Combining Eqs. (5) and (6) we obtain:

$$u_2^2 = \frac{2g \cdot (h - \Delta Z)}{\left[1 + k - \alpha_1 \cdot \left(\frac{d_2}{d_1} \right)^4 C_v^2 \right]}. \quad (7)$$

Since in our case d_1 is considerably larger than d_2 ($d_2/d_1 \approx 0.03$) and $C_v < 1$; expression (7) can be simplified to:

$$u_2^2 \cong \frac{2g}{[1+k]} (h - \Delta Z). \quad (8)$$

The implication of this expression is that, if we can measure the velocity of the exit jet u_2 as a function of h , according to our model, we should expect a linear relationship between

u_2^2 and h . In fact, this linear trend could be used to experimentally falsify our model. Furthermore, the slope of this line and its intersection with the axis could allow us to determine experimentally the values of the coefficients k and ΔZ .

From this same model, it is possible to determine the motion of the free surface (u_1) and the evacuation time t_e .⁶ Combining Eqs. (6) and (8) we have:

$$u_1 = -\frac{dh}{dt} = -C_v \left(\frac{d_2}{d_1} \right)^2 \sqrt{2g\mu} \sqrt{h - \Delta Z}, \quad (9)$$

where we have introduced the coefficient $\mu = 1/(1+k)$. The sign in this equation is related to the orientation chosen to define the positive direction of h and u_1 . Introducing a new constant A as:

$$A = C_v \left(\frac{d_2}{d_1} \right)^2 \sqrt{2g\mu}, \quad (10)$$

we can write a differential equation for h , which is:

$$\frac{dh}{\sqrt{h - \Delta Z}} = -A dt. \quad (11)$$

This expression can be easily integrated to give:

$$\frac{\sqrt{h - \Delta Z}}{\sqrt{h_0 - \Delta Z}} = \left(1 - \frac{t}{t_e} \right), \quad (12)$$

where h_0 is the height of the free surface for $t=0$ and t_e is the empty time, which, according to Eqs. (10) and (12), is given by:

$$t_e = \frac{\sqrt{h_0 - \Delta Z}}{C_v \left(\frac{d_2}{d_1} \right)^2 \sqrt{2g\mu}}. \quad (13)$$

Therefore, if we measure h as a function of time, if the model presented here is correct, the plot of $\sqrt{h - \Delta Z}$ as a function of t should be linear. The parameters of this straight line could provide us the value of t_e and allow us to find the value of C_v . For $t \approx t_e$, there is still some fluid above the orifice ($h \approx \Delta Z$), but there is no jet, the liquid just leaks out of the tank.

To determine the value of u_2 , we shall assume that the motion of the water particles that comprise the jet follow the same equation of motion as those of the horizontal projectile with initial velocity u_2 , so:

$$x(t) = u_2 t \quad \text{and} \quad y(t) = H - \frac{1}{2} g t^2, \quad (14)$$

where H is the height of the exit orifice relative to the origin of coordinate chosen, see figure 2. Combining these two expressions we obtain the equation for the trajectory of the jet:

$$y(x) = H - \frac{1}{u_2^2} \left(\frac{g}{2} x^2 \right). \quad (15)$$

Therefore, if we can determine experimentally the form of the trajectory of the exit jet of water, by fitting expression (15) to the experimental data, we could obtain the value of u_2 .

III. - Experiment

The experimental setup consisted of a transparent cylindrical tank with a lateral drainage orifice close to the bottom and a web cam connected to a computer. The vessel was 11 cm in diameter and 25 cm high, the diameter of the orifice was 3 mm and the wall thickness of the vessel was 2 mm. This type of vessel, with a horizontal capillary tube, has been used in the past to determine the viscosity of a fluid.⁸ The tank was filled with tap water with a few drops of blue ink to facilitate the visualization of the exit jet. The vessel was positioned in front of a board on which we had drawn a grid with lines every 10 cm to provide an absolute scale for the photographs. The exit jet was parallel to the gridded board. The web cam was placed just in front of the vessel, at about 1.5 m. In this manner we were able to photograph the jet of water with the gridded board in the background. Pictures were taken every time the free surface dropped by about 1 cm. Clearly a digital camera could also be used for this purpose. In this manner, each photograph recorded the height of water h in the cylinder and the trajectory of the exit jet of water from the orifice. To determine the trajectory of the jet it is possible to use almost any graphics program, such as Microsoft Photo Editor®. This program allows us to obtain the location (pixel coordinate) of any point in the picture. Furthermore, using the background grid, it is possible to transform the coordinate of any pixel in the photo to the grid coordinate. Another alternative is to use the picture as a background of a plot as described below.

The procedure we followed to obtain u_2 from the experimental data consisted in overlapping the digital photograph of the jet to a Microsoft Excel® graph with a transparent background.⁹ The grid of the Excel graph is set to coincide with the mesh grid used in the background of the picture. The graph is moved and stretched so that the mesh grid of the picture exactly coincides with the corresponding grid in the Excel graph. Once this condition is achieved, the origin (vertex) of the theoretical parabola, described by Equation (15), is chosen to coincide with the exit orifice. Then the value of u_2 is varied so that the theoretical curve described by Eq. (15) coincides with the experimentally observed trajectory. Figure 3 is an example that illustrates this procedure.

We also tracked the position of the free surface of the water in time. To facilitate the measurement of the height of liquid h , we drew horizontal marks every 0.5 cm, starting at the position of the drainage orifice. We used a stopwatch to measure the time it took for the free surface of the water to reach each horizontal mark. In this manner we measured h as a function of time.

IV - Results and Discussion

In figure 3.a) we present an actual digital photograph of the vessel and the exit jet. In figure 3.b) we show the same digital photograph of the exit jet in the background overlapped with a plot of the theoretical trajectory of the jet as described by expression (15). By adjusting the value of u_2 , we “fitted” the theoretical trajectory to the actual trajectory of the jet. The grid lines in the Excel plot were chosen to coincide with the grid lines in the background of the picture.⁹ This figure shows that the trajectory of the jet is very well reproduced by Eq. (15). This agreement is a clear indication that the liquid particles that conform the jet follow the same trajectory as the solid particles do.¹⁰ Therefore the liquid elements of volume are described by the same physical laws of

mechanics that govern the motion of solids. This result may be useful to confronting the Aristotelian misconception, still held by some students, that liquid and solids follow different laws. In figure 3 we have also included the trajectory of the jet that we would expect if there were no energy loss, Eq. (1); it is very clear that the naïve application of expression (1) gives a poor description of the data.

In figure 4, we present the plot of u_2^2 as a function of h . The fact that the relationship between u_2^2 and h is linear, indicates that the main assumptions of the model proposed, expressed by Eqs. (7) and (8), are in good agreement with the experimental results. Furthermore, by fitting the theoretical expectation, Eq. (8), to the data, we can obtain the values of the parameters ΔZ and k . In Table I we present the results of these parameters for different runs of the experiment. In figure 4 we also show the theoretical expectation that is obtained using expression (1), again we see that this approach give a poor description of the results. In this figure we also include a line, that would be obtained from Eq. (8) if we ignore the minor loss term ($k=0$), again the agreement of this approach with the data is poor. Therefore our experiment indicates that in order to reproduce our results, it is necessary to include two types of energy losses in Bernoulli's equation, one term dependent on the square of the average velocity (minor loss) and another term independent of the velocity (ΔZ). We performed several runs of the same experiment to test the consistency of our results and the robustness of the parameters of the model.

In figure 5 we present the results for the height of the free surface h as a function of time. To test the validity of our model, expression (12), we plotted the modified variable $\sqrt{(h - \Delta Z)/(h_0 - \Delta Z)}$ as a function of time. The fact that this plot has a clear linear trend indicates the validity of the model to describe the physics of the problem. In this figure we also include the theoretical expectation that we would obtain if we ignore the effect of vena contracta, i.e. $C_v=1$. We see that our results are inconsistent with this assumption and clearly indicate the necessity of taking in consideration the contraction of the jet. Furthermore, the parameter of the fitted line allows us to obtain the values of the empty time t_e and the coefficient of vena contracta C_v . In Table I we present the values of the relevant parameters of the model for several runs of the experiment. We notice that the values of the measured parameters are consistent for the different runs of the experiment. Furthermore, the value of k (coefficient of minor losses) and C_v are also consistent with the values reported in the literature for these parameters.^{1,2,3} There is a simple heuristic justification for the contraction of the jet suggested in the literature.^{2,11} The discharge jet has a momentum along the horizontal axis (x axis in figure 2), the rate of change of momentum is: $dp/dt = \rho \cdot Q \cdot u_2$. Here Q is the volume flow rate ($Q = C_v A \cdot u_2$) and A represents the outlet area. The force, responsible of this change of momentum, is associated to the hydrostatic pressure, $P = \rho g h$, i.e.:

$$PA = \rho g h A \approx C_v \cdot A \rho u_2^2. \quad (16)$$

Therefore, according to Eq. (8) we have:

$$C_v \approx \frac{1}{2} \cdot \left(\frac{1+k}{1-\Delta z/h} \right) \approx 0.59 < 1, \quad (17)$$

this simple argument provides us with a physical justification for the contraction of the jet and also gives a semi quantitative estimate of C_v . It is important to note that the contraction of the jet is also a consequence of the complex instabilities that take place around the orifice. The observed values of C_v depend, among other things, on the roundness of the edges of the aperture, the wall thickness, etc.¹ All these effects are clearly not accounted by our simplified argument discussed above.

In figure 6, on the vertical axis we plotted the value of the Reynolds number, Re , at the orifice for the different values of h used in our study. We see that in our experiment the value of Re varies between 4000 and nearly 400, indicating that the flow regime spans beginning of the turbulent, transitional and laminar flows. The fact that the model is able to reproduce the experimental data, using the same constants k , ΔZ and C_v , in all these regions of Re , indicates that hypothesis in our model, namely the type of losses proposed and the expanded Bernoulli's equation, are valid for these regimes of flows. It also follows from this analysis that the parameters of the model, namely k , ΔZ and C_v , are independent of Re in the region studied here.

V. - Conclusions

The experiment carried out in this study is simple and inexpensive. Also it is accessible to beginner students and it clearly illustrates the importance and usefulness of Bernoulli's equation for real fluids including energy losses, for a wide range of Reynolds numbers. The validity of the type of energy losses proposed and the expanded Bernoulli's equation, spans the beginning of the turbulent regime, as well as the transitional and laminar regimes. All that is needed to further explore the turbulent regime is a taller cylinder.

The experiment also verifies that liquid particles follow the same trajectory as solid particles, indicating that they obey the same physical laws.

The proposed model, based on the extended Bernoulli's equation, is adequate to describe qualitatively and quantitatively the physics of drainage of a vessel. Furthermore, the fitting of the theoretical model to the data allows us to extract the relevant parameters of the model, namely the coefficients ΔZ and k , of Eq. (4), and the coefficient of vena contracta C_v . Our results are consistent with values of these parameters that are independent of the Reynolds number in the region explored in this study.

The motion of the free surface is well reproduced by the model, in particular its dependence with time. Furthermore, this study allows us to measure the coefficient of vena contracta in the exit jet and the coefficient of losses in our system.

VI - Acknowledgment

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Appendix A

When a laminar flow enters into a pipe, it does not immediately develop the parabolic velocity profile that prevails well inside the pipe. There is a transition region or entrance

length^{1,3} (L_e) over which the velocity profile changes from a rather planar to the fully developed parabolic profile. There is a semi empirical relation that allows us to estimate this entrance length given by:¹²

$$L_e = C_e \cdot Re \cdot d, \quad (A1)$$

here $Re(=\rho \cdot u \cdot d / \eta)$ is the Reynolds number, d the inner diameter of the pipe and C_e is a constant. Several authors have proposed different values for this constant,^{1,3,12} but all range between 0.029 and 0.06 for laminar flows. If we apply this expression to the exit orifice, in all cases studied here, we obtained values of L_e that were much larger than the wall thickness. Therefore, the velocity profile of the exit jet can be regarded as planar in this case, i.e. $\alpha_2 \approx 1$. On the other hand, for the case of the cylinder, the average velocity is so small, ($u_1 \approx 10^{-2}$ cm/s) that the value of α_1 is approximately 2.

Parameter	Run 1	Run 2	Run 3
ΔZ (cm)	0.9 ± 0.2	0.9 ± 0.2	0.9 ± 0.2
μ	0.91 ± 0.01	0.90 ± 0.01	0.90 ± 0.01
k	0.09 ± 0.01	0.11 ± 0.01	0.11 ± 0.01
C_v	0.46 ± 0.03	0.47 ± 0.03	0.44 ± 0.03

Table I: Parameters obtained for several runs. The values of μ and C_v are consistent with those reported in the literature.^{1,2,3}

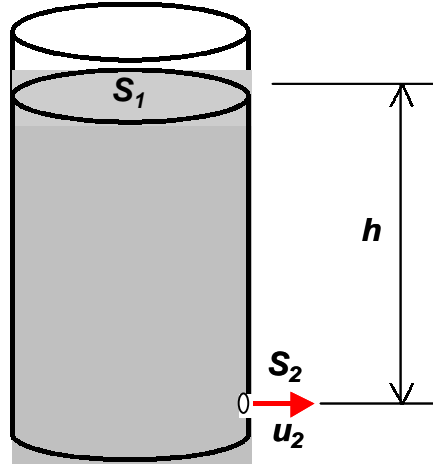


Figure 1: Schematic diagram of the vessel.

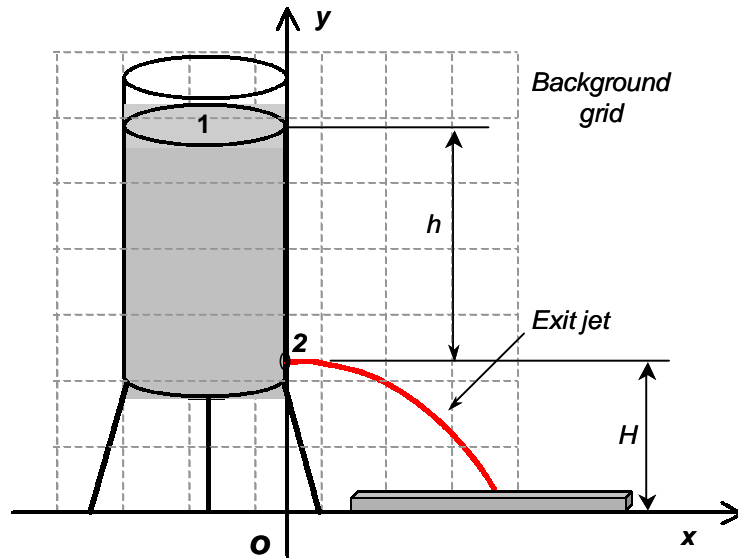


Figure 2: Schematic diagram of the experimental setup

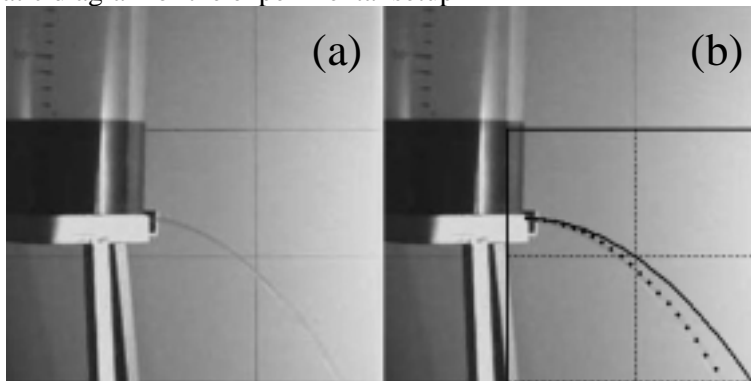


Figure 3: a) Actual digital photograph of the exit jet. b) The same photograph in the background overlapped with a plot of the theoretical trajectory (dotted line) of the jet as described by expression (15). The solid line represents the theoretical expectation if energy loss is completely ignored, Eq. (1).

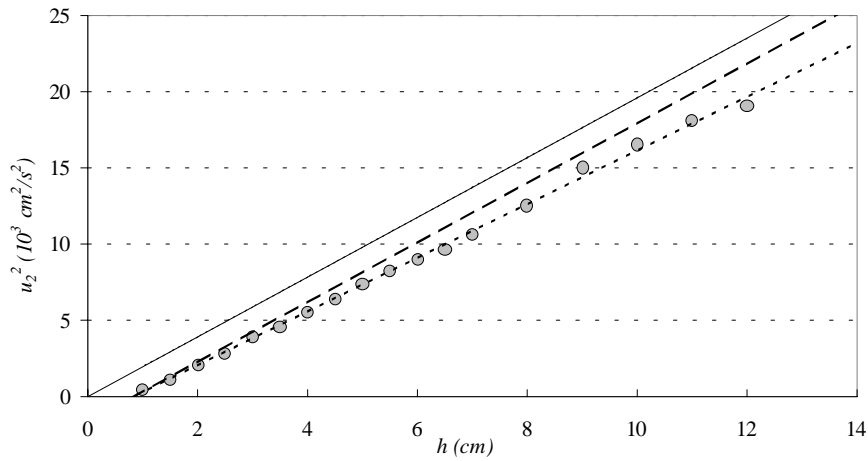


Figure 4: The circles represent the experimental results of u_2^2 as a function of the height h , the straight dotted line is a linear fit of the data using Eq. (8). The solid line represents the theoretical expectation if energy losses are completely ignored, Eq. (1). The dashed line is the expectation obtained ignoring the minor loss term ($k=0$).

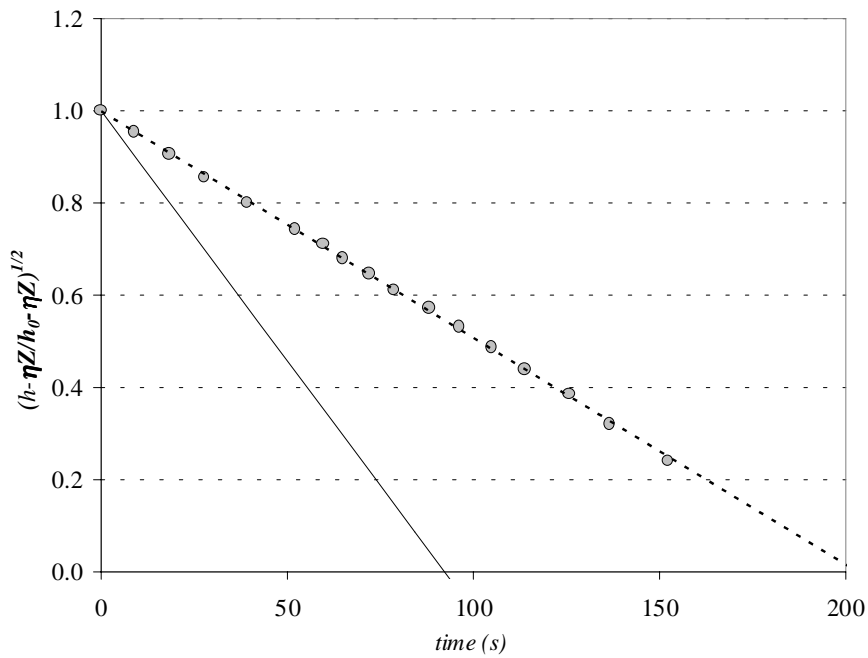


Figure 5: The circles represent the experimental results of $\sqrt{(h - \Delta Z)/(h_0 - \Delta Z)}$ as a function of time; the straight dotted line is a fit of the data. The solid line represents the theoretical expectation obtained disregarding the contraction of the jet ($C_v=1$).

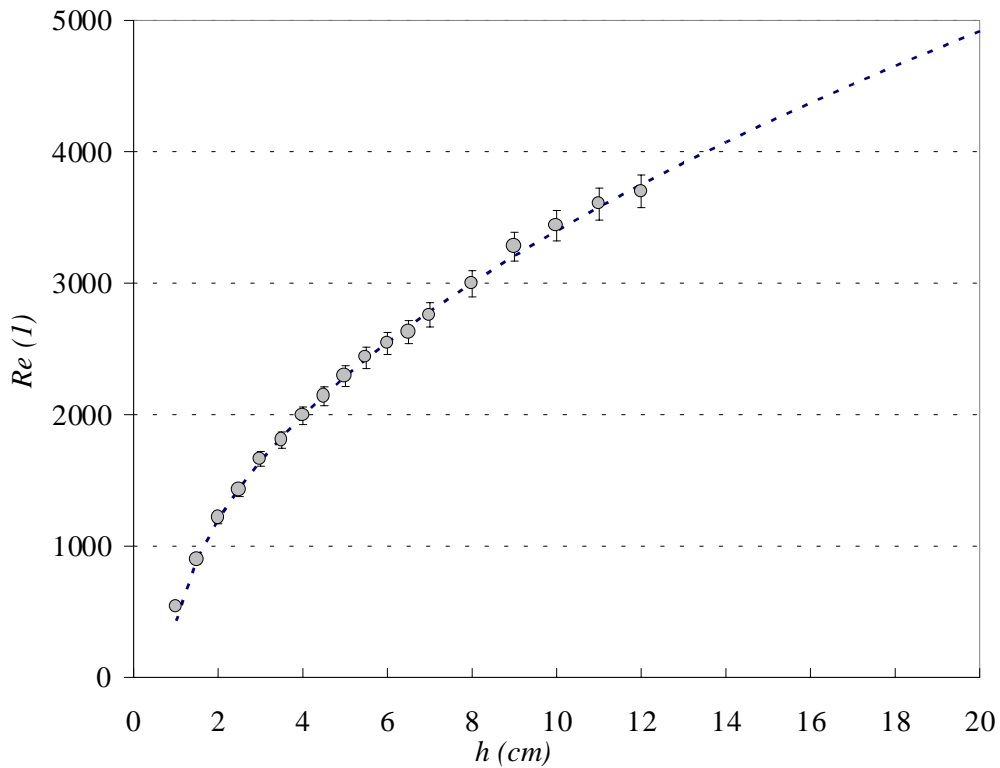


Figure 6: The square symbols indicate the values of the Reynolds number (right vertical scale) at the orifice for the different values of h used in our experiment. By choosing a taller cylinder it is possible to further explore turbulent regime.

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¹ B. R. Munson, D. F. Young and T. H. Okiishi, *Fundamentals of Fluid Mechanics*, (2nd edition J. Wiley & Son, New York, 1994), Chap. 3 and 8.

² V. L. Streeter and E. D. Wylie, *Fluid Mechanics*, 8th edition (Mc Graw-Hill, New York, 1985), Chap. 8.

³ B. Nekrasov, *Hydraulics for aeronautical engineers* (Translated from the Russian by V. Talmy), Moscow, Mir Publishers, 1969, Chap. 6 and 9.

⁴ C. E. Synolakis and H. S. Bader, "On combining the Bernoulli and Poiseuille equation- A plea to authors of college physics texts", *Am. J. Phys.* **57**, 1013 - 1019 (1989)

⁵ C. Waltham, S. Bendall and A. Kotlicki, "Bernoulli levitation", *Am. J. Phys.* **71**, 176 - 179 (2003)

⁶ J. N. Libii, "Mechanics of the slow draining of a large tank under gravity", *Am. J. Phys.* **71**, 1204 - 1207 (2003).

⁷ M. Nelkin, "Resource Letter TF-1: Turbulence in Fluids", *Am. J. Phys.* **68**, 310 - 318 (2000)

⁸ G. T. Hageseth "*Surface and kinetic energy densities: A fluid dynamics laboratory exercise*" Am. J. Phys. **54**, 1011 - 1014 (1986).

⁹ Examples of the Excel files illustrating this procedure can be downloaded from www.fisicareCreativa.com. Web site dedicated to experimental physics. It contains experimental projects using new technologies and reports of experiments performed by undergraduate students from different universities of Latin America.

¹⁰ B. Tolar "*The water drop parabola*", Phys. Teach. **18**, 371-372 (1980)

¹¹ R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics II*, Addison-Wesley, Reading, MA 1964 Vol.II Chap.41.

¹² S. G. Kandlikar and L. A. Campbell, "*Effect of entrance condition on frictional losses and transition to turbulence*", Proceedings of IMECE2002, ASME International Mechanical Engineering Congress & Exposition, November 17-22, 2002, Louisiana, New Orleans.